# SECTION 2.7: PRECISE DEFINITIONS OF LIMITS

**RECALL:** $\lim_{x\to a} f(x) = L$ means the outputs $f(x)$ get 'close' to $L$ as the inputs $x$ get 'close' to $a$.

**RHETORICAL QUESTION:** How 'close' is 'close'?

In this section, we aim to formalize the definition of limit. The definition we arrive at here is used to prove all of the limit laws we've grown accustomed to using. Central to our understanding of limit is the concept of 'distance' between real numbers, which we review presently.

**RECALL:** The distance between two real numbers $a$ and $b$ can be calculated using absolute value:

$$|a - b| = \text{the distance between the real numbers } a \text{ and } b$$

So, for example:

- The equation $|x - 3| = 1$ describes all real numbers $x$ that are 1 unit away from 3. (so $x = 2$ or $x = 4$.)
- The inequality $|x - 3| < 1$ describes all real numbers $x$ that are less than 1 unit away from 3. (so $2 < x < 4$.)

**EXAMPLE 1:** Use absolute values to describe all real numbers $x$ that are less than 0.1 units away from 3.

Since we'll be using absolute values extensively in our work here, let's review properties of the absolute value.

**PROPERTIES OF ABSOLUTE VALUE:** Suppose $a$, $b$, and $p$ are real numbers.

- **Product Rule:** $|ab| = |a||b|$
- **Power Rule:** $|a^p| = |a|^p$ provided $a^p$ is defined.
- **Quotient Rule:** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$, provided $b \neq 0$
- **TRIANGLE INEQUALITY:** $|a + b| \leq |a| + |b|$.
  **NOTE:** In general, $|a + b| \neq |a| + |b|$.

In a nutshell, the absolute value works well with multiplication, division, and powers.
It doesn't work so well with addition (or subtraction.)

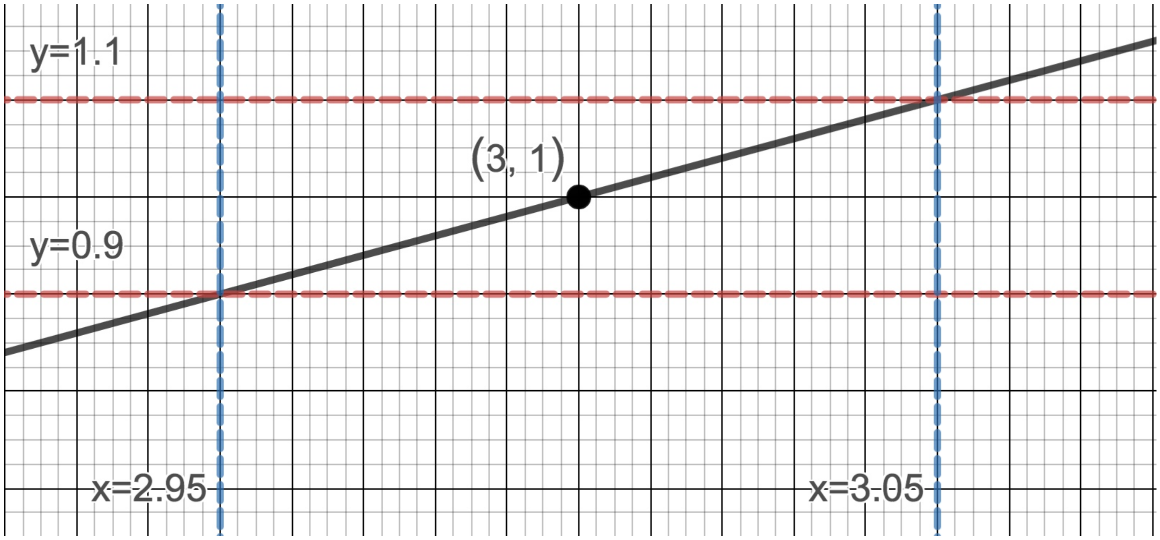
**EXAMPLE 2:** We know $\lim_{x\to 3}(2x - 5) = 1$. How close do the inputs $x$ have to be to 3 to guarantee the outputs from $f(x) = 2x - 5$ are less than 0.1 units away from the limit 1?

'$f(x) = 2x - 5$ is less than 0.1 units from 1' translates to the inequality: $|(2x - 5) - 1| < 0.1$.
Using properties of the absolute value, we get:

$$\begin{aligned}
|(2x-5)-1| &< 0.1 \\
|2x-6| &< 0.1 \\
|2(x-3)| &< 0.1 \\
|2||x-3| &< 0.1 && \text{product rule} \\
2|x-3| &< 0.1 \\
|x-3| &< \frac{0.1}{2} = 0.05
\end{aligned}$$

Hence, if $|x - 3| < 0.05$, that is, if $x$ is less than 0.05 units from 3, $f(x) = 2x - 5$ should be less than 0.1 units away from 1. To prove this algebraically, all we need to do is reverse our steps in our work above.

Let's see what this all looks like graphically. For $f(x) = 2x - 5$ to be within 0.1 units of 1 means the $y$-values on the graph of $y = f(x)$ must lie between the horizontal lines $y = 1 - 0.1$ or $y = 0.9$ and $y = 1 + 0.1$ or $y = 1.1$. Dropping down to the $x$-axis, this means $x$ must lie between $x = 2.95$ and $x = 3.05$, or within 0.05 units of 3.




Taking a step back, we note that the graph of $f(x) = 2x - 5$ is a line with slope $m = \dfrac{\Delta y}{\Delta x} = 2$.

Rearranging, we get: $\Delta x = \dfrac{\Delta y}{2}$. So, if $\Delta y < 0.1$ units, then $\Delta x = \dfrac{\Delta y}{2} < \dfrac{0.1}{2} < 0.05$ units, and vice-versa.

**EXAMPLE 3:** Once again looking at: $\lim_{x \to 3}(2x - 5) = 1$. How close do the inputs $x$ have to be to 3 to guarantee the outputs from $f(x) = 2x - 5$ are less than 0.01 units away from the limit 1?
Determine your answer algebraically, and verify your answer graphically.

We could keep going: how close does $x$ have to be to 3 to guarantee $f(x) = 2x - 5$ is less than **0.001** units away from 1? less than **0.0001** units away from 1? less than less than $\mathbf{1.6 \times 10^{-35}}$ units away from 1?
Since there is no 'closest' number to 1, we need to make sure that **whatever** tolerance we're given, no matter **how close** we want $f(x) = 2x + 5$ to be to 1, we can **always** get $x$ close enough to 3 to do the job.

**INTRODUCING $\epsilon$ and $\delta$:**

- Let's let the Greek letter epsilon, $\epsilon$, denote how close we want $f(x) = 2x - 5$ to be to 1. Note, $\epsilon > 0$.
- Let's let the Greek letter (lowercase) delta, $\delta$, denote how close $x$ needs to be to 3 to guarantee $f(x)$ is less than $\epsilon$ units away from 1. Note, $\delta > 0$ as well.

Working through the algebra, we get:

$$\begin{aligned}
|(2x - 5) - 1| &< \epsilon \\
|2x - 6| &< \epsilon \\
|2(x - 3)| &< \epsilon \\
|2||x - 3| &< \epsilon \quad \text{product rule} \\
2|x - 3| &< \epsilon \\
|x - 3| &< \frac{\epsilon}{2}
\end{aligned}$$

Hence, if we set $\delta = \dfrac{\epsilon}{2}$, then if $|x - 3| < \dfrac{\epsilon}{2}$, then the work above shows $f(x) = 2x - 5$ will be within $\epsilon$ units of 1.

We are now in position to state the formal (precise) definition of limit:

**DEFINITION:** $\lim_{x \to a} f(x) = L$ means given $\epsilon > 0$ there is a $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

In English, the definition says to prove $\lim_{x \to a} f(x) = L$ means we have to show that we can get $f(x)$ as close as we like (within '$\epsilon$' units) of $L$ by getting $x$ close enough (within '$\delta$' units) of $a$.

**NOTE:** The '0' in the inequality $0 < |x - a| < \delta$ is there because don't actually care what's happening **at** $x = a$. In the previous example, $f(x) = 2x - 5$ is continuous at $x = 3$, so excluding what was happening at 3 wasn't needed. In general, though we need $0 < |x - a| < \delta$ since $f(a)$ may be undefined or $f(a)$ may exist but $f(a) \neq L$.

**EXAMPLE 4:** Using our work above, we can write a **formal proof** that $\lim_{x \to 3}(2x - 3) = 5$ as follows:

Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{2}$. If $0 < |x - 3| < \frac{\epsilon}{2}$, then: $|(2x - 5) - 1| = |2x - 6| = |2(x - 3)| = |2||x - 3| < 2\left(\frac{\epsilon}{2}\right) = \epsilon$.

**NOTE:** All the 'formal proof' really does is rephrase our 'scratchwork.'

**EXAMPLE 5:** Use the precise definition of limit to prove $\lim_{x \to -2} \frac{4 - x}{3} = 2$.

**STEP ONE: SCRATCHWORK:** We start with solving: $|f(x) - L| < \epsilon$, or, in this case: $\left|\frac{4 - x}{3} - 2\right| < \epsilon$.

We want to transform this equation into something $|x - a| < \delta$, or in this case, $|x - (-2)| = |x + 2| < \delta$.

$$
\begin{aligned}
\left|\frac{4 - x}{3} - 2\right| &< \epsilon \\
\left|\frac{4 - x}{3} - \frac{6}{3}\right| &< \epsilon \\
\left|\frac{4 - x - 6}{3}\right| &< \epsilon \\
\left|\frac{-x - 2}{3}\right| &< \epsilon \\
\left|-\frac{1}{3}(x + 2)\right| &< \epsilon \\
\left|-\frac{1}{3}\right| |x + 2| &< \epsilon \\
\frac{1}{3}|x + 2| &< \epsilon \\
|x + 2| &< 3\epsilon
\end{aligned}
$$

**STEP TWO: FORMAL PROOF:**

Given $\epsilon > 0$, choose $\delta = 3\epsilon$. If $0 < |x + 2| < 3\epsilon$, then: $\left|\frac{4 - x}{3} - 2\right| = \cdots = \frac{1}{3}|x + 2| < \left(\frac{1}{3}\right)(3\epsilon) = \epsilon \checkmark$.

**EXAMPLE 6 (VIDEO):** Use the precise definition of limit to prove the following.

1. $\lim_{x\to 2}(5x+1) = 11$

   Ans:
   Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{5}$. Then if $0 < |x-2| < \frac{\epsilon}{5}$,
   $$|(5x+1)-11| = |5x-10| = |5(x-2)| = 5|x-2| < 5\left(\frac{\epsilon}{5}\right) = \epsilon \checkmark$$

2. $\lim_{x\to -2}(4x-1) = -9$

   Ans:
   Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{4}$. Then if $0 < |x+2| < \frac{\epsilon}{4}$,
   $$|(4x-1)+9| = |4x+8| = |4(x+2)| = 4|x+2| < 4\left(\frac{\epsilon}{4}\right) = \epsilon \checkmark$$

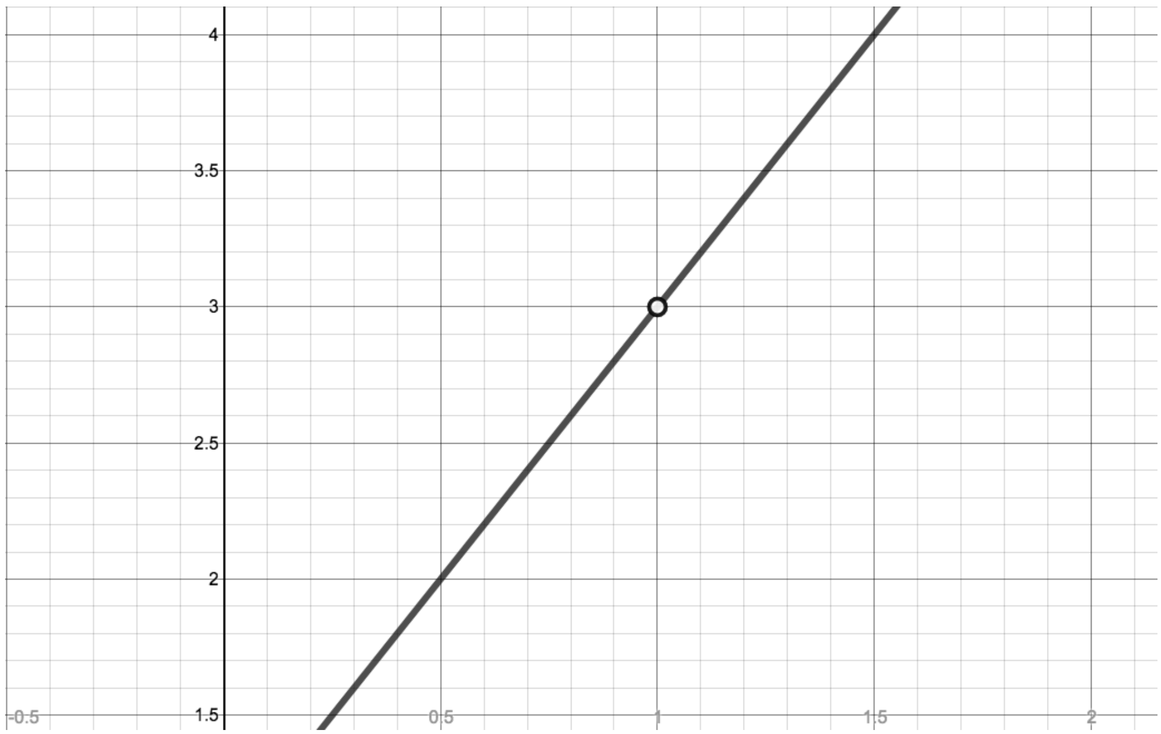
3. $\lim_{x\to -1}(4-2x) = 6$

   Ans:
   Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{2}$. Then if $0 < |x+1| < \frac{\epsilon}{2}$,
   $$|(4-2x)-6| = |-2-2x| = |(-2)(x+1)| = 2|x+1| < 2\left(\frac{\epsilon}{2}\right) = \epsilon \checkmark$$

4. $\lim_{x\to -\frac{1}{3}} \frac{5-3x}{2} = 3$

   Ans:
   Given $\epsilon > 0$, choose $\delta = \frac{2}{3}\epsilon$. Then if $0 < \left|x+\frac{1}{3}\right| < \epsilon$,
   $$\left|\frac{5-3x}{2}-3\right| = \left|\frac{-1-3x}{2}\right| = \left|-\frac{3}{2}\left(x+\frac{1}{3}\right)\right| = \frac{3}{2}\left|x+\frac{1}{3}\right| < \frac{3}{2}\left(\frac{2}{3}\epsilon\right) = \epsilon \checkmark$$

**EXAMPLE 7:** For the scenario below, $\lim_{x\to 1} f(x) = 3$.

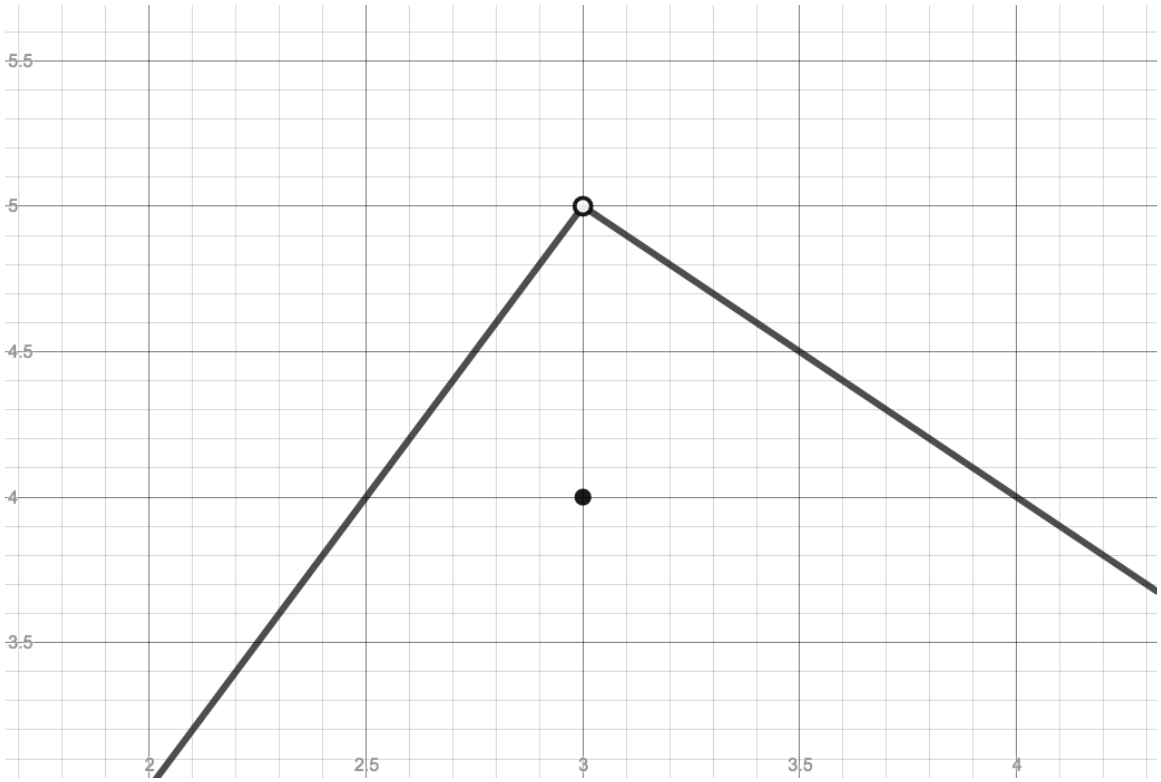



Estimate $\delta$ so that if $0 < |x-1| < \delta$, then

- $|f(x) - 3| < 1$
- Ans: $0 < \delta \leq 0.5$
- $|f(x) - 3| < 0.5$
- Ans: $0 < \delta \leq 0.25$
- $|f(x) - 3| < 0.1$
- Ans: $0 < \delta \leq 0.05$

Make a conjecture on the relationship between $\epsilon$ and $\delta$.

**EXAMPLE 8:** For the scenario below, $\lim_{x\to 3} f(x) = 5$.

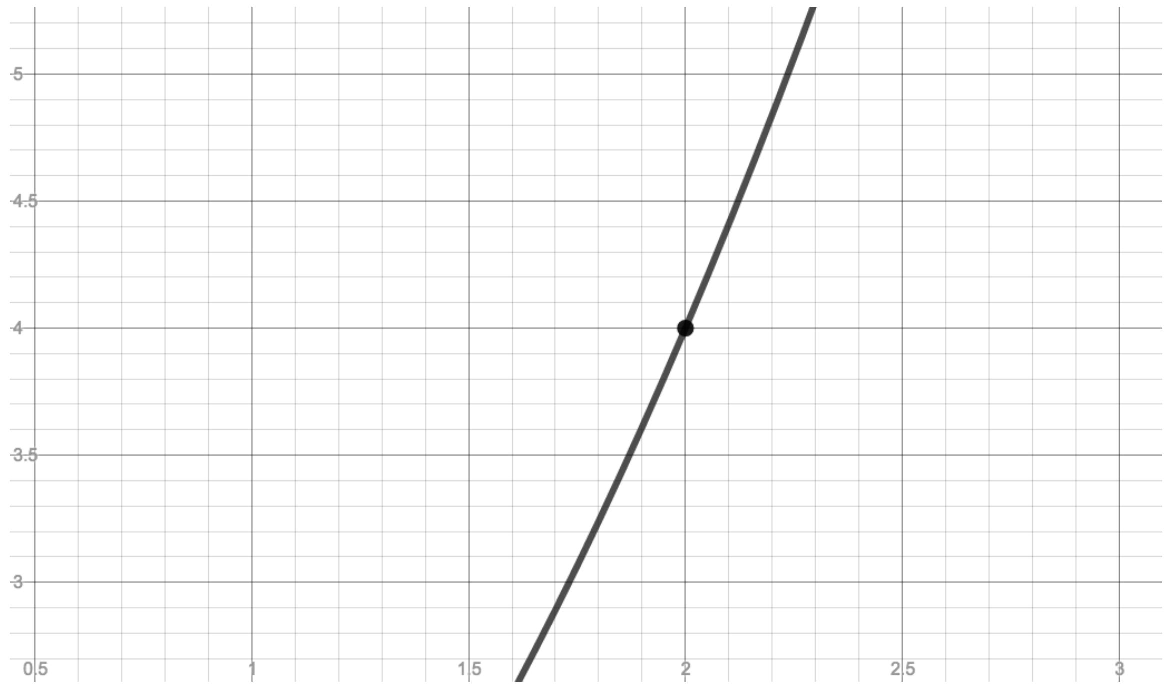



Estimate $\delta$ so that if $0 < |x-3| < \delta$, then

- $|f(x) - 5| < 1$
- Ans: $0 < \delta \le 0.5$
- $|f(x) - 5| < 0.5$
- Ans: $0 < \delta \le 0.25$
- $|f(x) - 5| < 0.1$
- Ans: $0 < \delta \le 0.05$

How was finding the $\delta$ in this problem different than in the last problem?

**EXAMPLE 9:** For the scenario below, $\lim_{x\to 2} f(x) = 4$.




Estimate $\delta$ so that if $|x-2| < \delta$, then

- $|f(x) - 4| < 1$
- Ans: $0 < \delta \le 0.2$ (approx.)

- $|f(x) - 4| < 0.5$
- Ans: $0 < \delta \le 0.1$ (approx.)

- $|f(x) - 4| < 0.1$
- Ans: $0 < \delta \le 0.01$ (approx.)

What difficulties do you encounter?

As the last two problems illustrate, estimating a $\delta$ for a given $\epsilon$ can be problematic for non-linear functions. The last example was none other than the graphical view of $\lim_{x \to 2} x^2 = 4$, which we tackle analytically below.

**EXAMPLE 10:** Use the precise definition of limit to prove $\lim_{x \to 2} x^2 = 4$.

**STEP ONE: SCRATCHWORK:**

As usual, we start with $|f(x) - L| = \left|x^2 - 4\right| < \epsilon$ and keep an eye out for something of the form $|x - a| = |x - 2| < \delta$:

$$\begin{aligned} \left|x^2 - 4\right| &< \epsilon \\ |(x+2)(x-2)| &< \epsilon \\ |x+2||x-2| &< \epsilon \end{aligned}$$

If we proceed as before, we'd divide both sides of the inequality $|x+2|$ ... but then we'd be trying to bound $x$ by an expression involving '$x$.' (Do you see why this would be problematic?)

Remember that our ultimate goal here is to make $\left|x^2 - 4\right| = |x+2||x-2|$ very small as $x$ gets close to 2.

Instead of trying to consider all real numbers $x$, why not restrict our attention to $x$ values close to 2.

How close to 2? Let's say within 1 unit of 2. These $x$ values satisfy $|x - 2| < 1$ or, equivalently, $1 < x < 3$.

If $1 < x < 3$, then $|x+2| < 5$, which makes $\left|x^2 - 4\right| = |x+2||x-2| < 5|x-2|$. Solving $5|x-2| < \epsilon$ gives $|x-2| < \frac{\epsilon}{5}$.

Hence, if $|x-2| < 1$ **and** $|x-2| < \frac{\epsilon}{5}$, then:

$$\left|x^2 - 4\right| = |(x+2)(x-2)| = |x+2||x-2| < 5|x-2| < 5\left(\frac{\epsilon}{5}\right) = \epsilon \checkmark$$

The way we guarantee $|x-2| < 1$ **and** $|x-2| < \frac{\epsilon}{5}$ is to choose $\delta$ to be the **smaller** of 1 and $\frac{\epsilon}{5}$. (Do you see why?)

**STEP TWO: FORMAL PROOF:** Given $\epsilon > 0$, choose $\delta$ to be the smaller of 1 and $\frac{\epsilon}{5}$.

If $0 < |x-2| < \delta$, then $\left|x^2 - 4\right| = |x+2||x-2| < 5|x-2| < 5\left(\frac{\epsilon}{5}\right) = \epsilon \checkmark$

**EXAMPLE 11 (VIDEO):** Use the precise definition of limit to prove $\lim_{x \to 3} x^2 = 9$.

Given $\epsilon > 0$, choose $\delta$ to be the smaller of 1 and $\frac{\epsilon}{7}$. Then if $|x-3| < \delta$: $|x^2 - 9| = |x+3||x-3| < 7|x-3| < 7\left(\frac{\epsilon}{7}\right) = \epsilon \checkmark$

Below we remind you of the precise definition of $\lim_{x \to a} f(x) = L$ we developed earlier:

**DEFINITION:** $\lim_{x \to a} f(x) = L$ means given $\epsilon > 0$ there is a $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

We can tweak the precise definition of limit to precisely define other limit concepts in this chapter. Compare and contrast each of the definitions below to the precise definition of limit given above. Do you see how the modifications to the definition above get at the concept being defined?

- $\lim_{x \to a^+} f(x) = L$ means given $\epsilon > 0$ there is a $\delta > 0$ so that if $a < x < a + \delta$ then $|f(x) - L| < \epsilon$.

- $\lim_{x \to a^-} f(x) = L$ means given $\epsilon > 0$ there is a $\delta > 0$ so that if $a - \delta < x < a$ then $|f(x) - L| < \epsilon$.

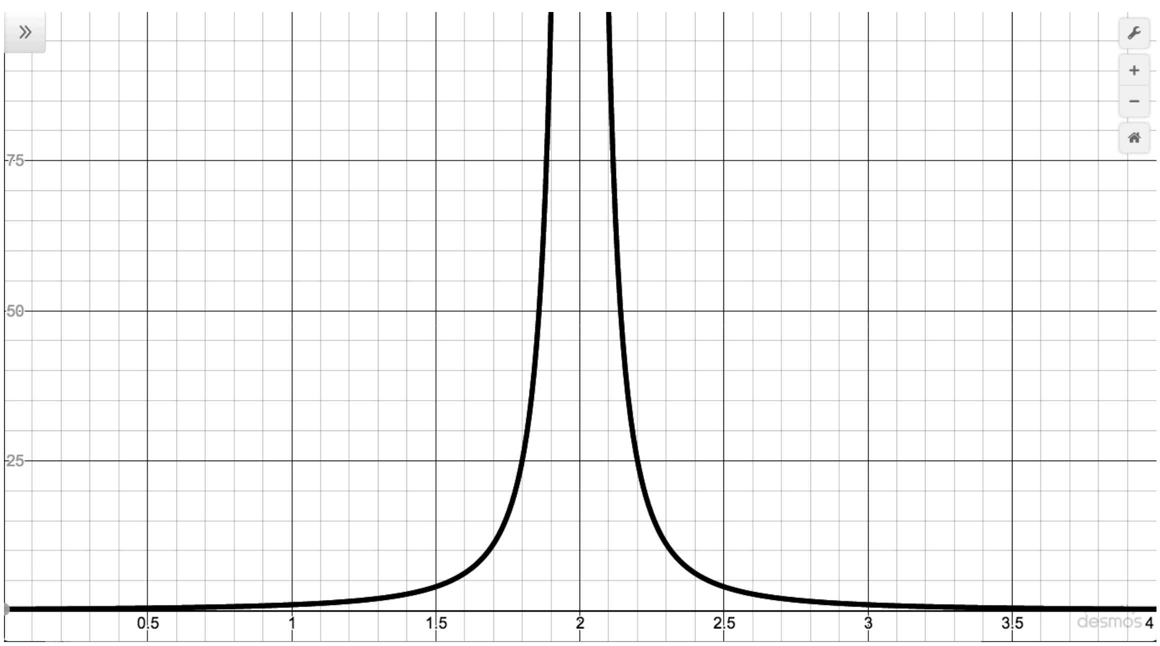
- $f$ is continuous at $x = a$ means given $\epsilon > 0$ there is a $\delta > 0$ so that if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

**PRECISE DEFINITION OF INFINITE LIMITS:**

- $\lim_{x \to a} f(x) = \infty$ means given $M > 0$ there is a $\delta > 0$ so that if $0 < |x - a| < \delta$ then $f(x) > M$.

- $\lim_{x \to a} f(x) = -\infty$ means given $M < 0$ there is a $\delta > 0$ so that if $0 < |x - a| < \delta$ then $f(x) < M$.

**EXAMPLE 12:** For the scenario below, $\lim_{x \to 2} f(x) = \infty$.




Use the graph to estimate $\delta$ so that if $0 < |x - 2| < \delta$:

- $f(x) > 25$
- Ans: $0 < \delta \le 0.2$
- $f(x) > 50$
- Ans: $0 < \delta \le 0.125$
- $f(x) > 75$
- Ans: $0 < \delta \le 0.1$

**EXAMPLE 13: (VIDEO)** Prove $\lim_{x\to 3} \frac{4}{(x-3)^2} = \infty$.

**SCRATCHWORK:** We want $\frac{4}{(x-3)^2} > M$. Rewriting, $(x-3)^2 < \frac{4}{M}$. Hence, $\sqrt{(x-3)^2} < \sqrt{\frac{4}{M}}$ or $|x-3| < \frac{2}{\sqrt{M}}$.

**FORMAL PROOF:** Let $M > 0$ be given. Let $\delta = \frac{2}{\sqrt{M}}$. If $0 < |x-3| < \frac{2}{\sqrt{M}}$, then $\frac{4}{(x-3)^2} > 4\left(\frac{\sqrt{M}}{2}\right)^2 = M$ ✓

**PRECISE DEFINITIONS OF LIMITS AT INFINITY:**

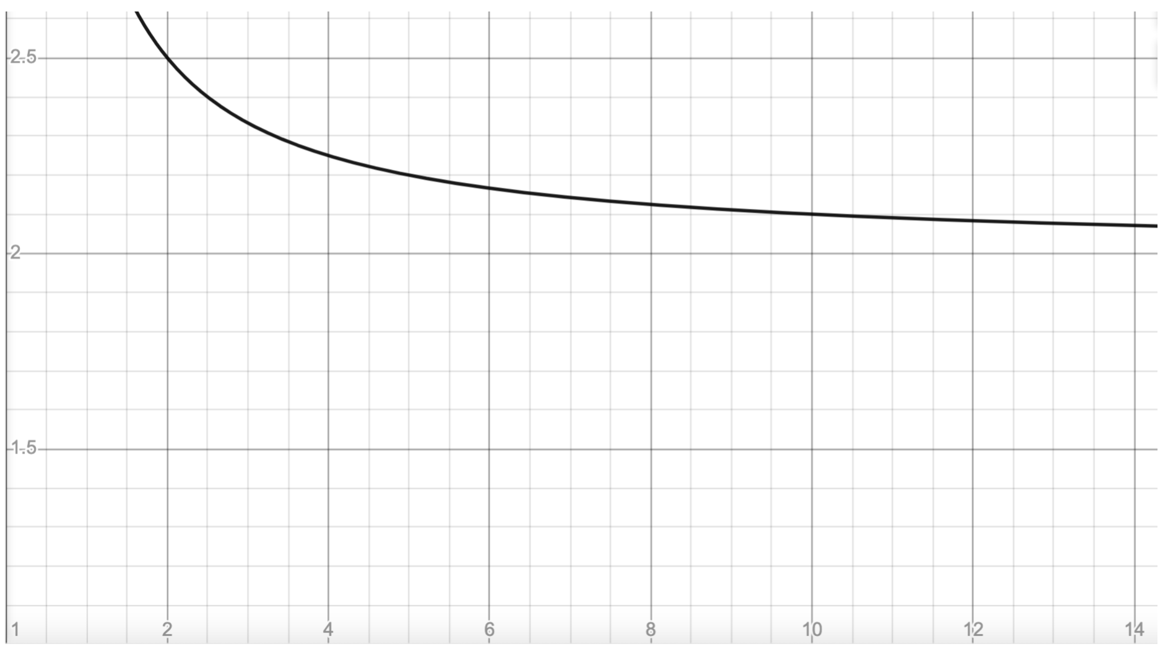
- $\lim_{x\to\infty} f(x) = \infty$ means given any number $M > 0$, there is a number $N$ so that if $x > N$, $f(x) > M$.

  **QUESTION:** How would you modify the verbiage above to define $\lim_{x\to\infty} f(x) = -\infty$? $\lim_{x\to-\infty} f(x) = \infty$? $\lim_{x\to-\infty} f(x) = -\infty$?

- $\lim_{x\to\infty} f(x) = L$ means given $\epsilon > 0$, there is a number $N$ so that if $x > N$, $|f(x) - L| < \epsilon$.

  **QUESTION:** How would you modify the verbiage above to define $\lim_{x\to-\infty} f(x) = L$?

**EXAMPLE 14:** For the scenario below, $\lim_{x\to\infty} f(x) = 2$.




Use the graph to estimate $N$ so that if $x > N$:

- $|f(x) - 2| < 0.5$
- Ans: $N \geq 2$
- $|f(x) - 2| < 0.25$
- Ans: $N \geq 4$
- $|f(x) - 2| < 0.1$
- Ans: $N \geq 10$

**EXAMPLE 15:** Prove $\lim_{x\to\infty} x^2 = \infty$.

**SCRATCHWORK:** We want $x^2 > M$. Extracting square roots, we get: $\sqrt{x^2} > \sqrt{M}$ so $|x| > \sqrt{M}$. Since $x \to \infty$, $|x| = x$ so we have that $x^2 > M$ is equivalent to $x > \sqrt{M}$.

**FORMAL PROOF:** Let $M > 0$ be given. Let $N = \sqrt{M}$. If $x > N$, then $x^2 > N^2 = \left(\sqrt{M}\right)^2 = M$ ✓

**EXAMPLE 16 (VIDEO):** Prove $\lim_{x\to\infty} \frac{3}{x} = 0$.

**SCRATCHWORK:** Given $\epsilon > 0$, we want: $\left|\frac{3}{x} - 0\right| < \epsilon$ or, in other words, $\frac{3}{|x|} < \epsilon$.

Since $x \to \infty$, $|x| = x$ so $\frac{3}{|x|} < \epsilon$ is equivalent to $\frac{3}{x} < \epsilon$ or $x > \frac{3}{\epsilon}$.

**FORMAL PROOF:** Given $\epsilon > 0$, choose $N = \frac{3}{\epsilon}$. Then if $x > N$, $\left|\frac{3}{x} - 0\right| = \frac{3}{|x|} = \frac{3}{x} < \frac{3}{N} = 3\left(\frac{\epsilon}{3}\right) = \epsilon$ ✓

**HOMEWORK:** Section 2.7: 9, 11, 15, and HANDOUT